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APPLICATION OF HYPERBOLIC ANALYSIS TO THE DISCHARGE OF A CONDENSER.

BY ALEXANDER MACFARLANE.

In recent years the theory of the discharge of an electric condenser has played a very important part in the advance of electrical science; for it served as the starting point of the experiments of Feddersen, Paalzow, Helmholz, Lodge, Hertz and many others, which culminated in the demonstration of the existence and properties of electromagnetic waves. The theory of the discharge was first given by Lord Kelvin, then Professor William Thomson, in a paper on "Transient Electric Currents" published in the June number of the *Philosophical Magazine* for 1853. The application to the phenomenon of the principle of the conservation of energy leads to the differential equation

$$\frac{d^2 q}{d t^2} + \frac{R}{L} \frac{d q}{d t} + \frac{1}{L C} q = 0 \qquad (1)$$

where R denotes the resistance and L the inductance of the circuit, and C the capacity of the condenser which is practically the capacity of the whole circuit. If $q = A e^{mt}$ be assumed as the solution of the equation, then m must be such that

$$A e^{mt} \left(m^2 + \frac{R}{L} m + \frac{1}{L C} \right) = 0$$

which reduces to

$$m^2 + 2 a m + b = 0 \qquad (2)$$

where for brevity a is written for $\frac{R}{2L}$ and b for $\frac{1}{LC}$.

According to the theory of the quadratic equation, there are two general cases separated by a transition case. If a^2 is greater than b; there are two real values of m, namely

$$-a + \sqrt{a^2 - b} \text{ and } -a - \sqrt{a^2 - b}.$$

If a^2 is less than b, there are two imaginary values of m, namely,

$$-a + \sqrt{-1}\,\sqrt{b - a^2} \text{ and } -a - \sqrt{-1}\,\sqrt{b - a^2}.$$

The transition or separating case is where $a^2 = b$: then there is only one value for m, namely, what is common to the two general values.

The following are the solutions which are usually given of the differential equation. In the case of real roots

$$q = c_1\, e^{-(a - \sqrt{a^2 - b})t} + c_2\, e^{-(a + \sqrt{a^2 - b})t}; \tag{3}$$

in the case of imaginary roots,

$$q = c_1\, e^{-(a - \sqrt{-1}\,\sqrt{b - a^2})t} + c_2\, e^{-(a + \sqrt{-1}\,\sqrt{b - a^2})t}; \tag{4}$$

and in the transition case

$$q = e^{-at}\,(c_1 + c_2\, t). \tag{5}$$

In the imaginary case, the apparently impossible solution is reduced to the form

$$q = A\, e^{-at} \sin\left[\sqrt{(b - a^2)}\; t + \varphi\right] \tag{6}$$

which shows that the change in the condenser at any time is given by a sine wave of period $\dfrac{2\pi}{\sqrt{b - a^2}}$ and of amplitude which diminishes geometrically at the rate a.

As the limiting case separates the two complementary regions of the real and the imaginary, we expect that the real solution is also capable of reduction to a form analogous to (6) and exhibiting the function with equal clearness. We also expect the transition solution to be evident from the two general solutions; but when they are in the above forms, the transition is not evident. We observe that in the former general case the roots are treated as simple algebraic quantities, while in the latter general case they are treated as complex quantities. A complex quantity consists of two components, one of which is real and the other imaginary. If there is any thorough going analogy, it must be possible to treat the real roots also as a species of complex quantity.

A complex quantity $a + b\,\sqrt{-1}$ can be reduced to the form

$r\,(\cos\theta + \sqrt{-1}\sin\theta)$; for $r = \sqrt{a^2+b^2}$, $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$. If we enquire into the geometrical meaning of the $\sqrt{-1}$ here appearing, we shall find that it means a quadrant of turning round the axis perpendicular to the plane of reference. Let β denote that axis, then β^θ denotes an angle of θ radians round the axis β, and

$$\beta^\theta = \cos\theta + \sin\theta\, \beta^{\frac{\pi}{2}}.$$

Hence the ordinary complex quantities can be expressed in the form

$$r\,\beta^\theta = r\,(\cos\theta + \sin\theta\,\beta^{\frac{\pi}{2}}),$$

and they are simply coaxial quaternions, the axis being commonly left unspecified, as it is the same for all.

Let $s\,\beta^\varphi$ denote another complex quantity, than $r\,\beta^\theta \times s\,\beta^\varphi = rs\,\beta^{\theta+\varphi}$

$$= r\,s\,[\cos\theta\cos\varphi - \sin\theta\sin\varphi + (\cos\varphi\sin\theta + \cos\theta\sin\varphi)\,\beta^{\frac{\pi}{2}}].$$

Here the product is formed according to the theorem for the cosine and the sine of the sum of two circular angles. Now the circular trigonometry has its complete counterpart in the hyperbolic trigonometry; consequently we expect to find a hyperbolic complex number. This subject was investigated at length in "Papers on Space Analysis,"[1] which I published 1891 to 1894. In this paper I propose to show that by treating the real root as a hyperbolic complex quantity, equation (3) can be reduced in precisely the same way as equation (4).

The exponential expression for a circular angle x is $e^{\sqrt{-1}\,x}$, which expressed definitely is $e^{x\,\beta^{\frac{\pi}{2}}}$. By applying the exponential theorem, we obtain a series which breaks up into two parts, namely,

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} +,$$

and

$$\sqrt{-1}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} -\right],$$

of which the former is the series for $\cos x$ and the latter the series for $\sin x$. Now because the terms of the sine series are all affected by the sign $\sqrt{-1}$, they do not add directly to the other

1. "The Imaginary of Algebra." *Proc.* A. A. A. S. vol. —, p. 50. Fundamental Theorems of Analysis, p. 28. Definitions of the Trigonometric Functions, p. 30. Principles of Elliptic and Hyperbolic Analysis, p. 17.

terms, but are geometrically compounded as forming a perpendicular component to the terms of the cosine series. We enquire for the analogous exponential expression for a hyperbolic angle x. Algebra furnishes none. It is not e^x, for

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} +$$

and here there is no ground for breaking up the series into two components; all the terms are real, and so add directly. For the same reason, it cannot be e^{-x}. But we know that

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} +,$$

$$\text{and } \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ;$$

there must therefore be some proper way of expressing the sum by an exponential function.

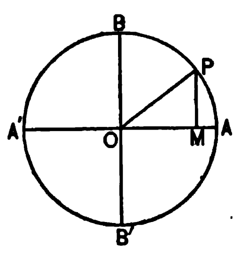

FIG. 1.

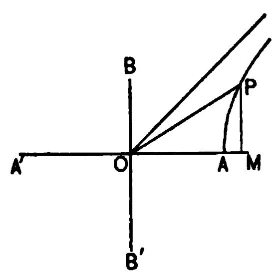

FIG. 2.

Before proceeding further, let us consider what is meant by a hyperbolic angle.

In Fig. 2, let A P be an arc of an equilatoral hyperbola, O A and O B the equal semi-axes. The radius O P is derived from the semi-axis O A by a hyperbolic versor which has a magnitude x and an axis through O perpendicular to the plane. Now x is not the ratio of the arc A P to either the radius vector O P or the semi-axis O A; but the ratio of twice the area of the sector A O P to the square on O A. In the circle, Fig. 1, the ratio of twice the area of the sector A O P to the square on O A is equal to that of the arc A P to the semi-axis O A; the symbol x may denote either. But in the hyperbolic counterpart it is the ratio of the areas which must be taken. If x denotes the ratio of twice the area of the hyperbolic sector A O P to the square on O A, then as a matter of truth, not mere definition, cosh x, by which is meant the ratio of O M to O A, is equal to

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+;$$

and sinh x, by which is meant the ratio of M P to O A, is equal to

$$x+\frac{x^3}{3!}+\frac{x^5}{5!}+ \quad .$$

We observe that O M and O A have the same direction, while M P s at right angles to O A; hence we conclude that the second series is really at right angles to the first. But instead if $\cos^2 x + \sin^2 x = 1$, we have $\cosh^2 x - \sinh^2 x = 1$; the fact that it is the difference not the sum of the squares which is equal to 1 attaches a scalar $\sqrt{-1}$ before the sinh series. We conclude that the proper expression for the hyperbolic versor is

$$\cosh x + \sqrt{-1} \sinh x \, \beta^{\frac{\pi}{2}};$$

and that the exponential expression is $e^{\sqrt{-1}\, x \, \beta^{\frac{\pi}{2}}}$. For brevity we will donate $\beta^{\frac{\pi}{2}}$ by i. Thus e^{ix} denotes a circular angle, and $e^{\sqrt{-1}\, i x}$ a hyperbolic angle.

The process by which equation (4) is usually reduced to equation (6) is highly obscure to the student. We shall state it in a form, such that it will apply to the analogous hyperbolic case. For brevity let n denote the square root of the difference of a^2 and b; in the hyperbolic case n is less than a. Equation (4) may then be written

$$q = e^{-at}\left(c_1\, e^{int} + c_2\, e^{-int}\right).$$

The arbitrary constants c_1 and c_2 are circular complex quantities; they are not perfectly arbitrary, but are connected in such a way that they involve only two independent quantities. Their magnitudes are equal and their angles supplementary. Hence we can write:

$$c_1 = c\,(\cos\varphi + i \sin\varphi),$$
$$c_2 = c\,(-\cos\varphi + i \sin\varphi);$$

then:

$$q = 2\,c\,e^{-at}\left(\cos\varphi\,\frac{e^{int}-e^{-int}}{2} + i\sin\varphi\,\frac{e^{int}+e^{-int}}{2}\right)$$
$$= i\,2\,c\,e^{-at}\left(\cos\varphi \sin nt + \sin\varphi \cos nt\right)$$
$$= i\,2\,c\,e^{-at}\sin(nt+\varphi).$$

The i is dropped, $2c$ is written A, and thus equation (6) is obtained.

The assumptions usually made in reducing are

$$c_2 = c\,(\cos\varphi + i\sin\varphi) \text{ and } c_2 = c\,(\cos\varphi - i\sin\varphi)$$

which is equivalent to making the angles conjugate. The solution then is

$$q = 2\,c\,e^{-at}\cos\,(nt + \varphi)$$

which is the horizontal instead of the vertical projection. The analogous investigation shows that the former is the correct assumption for the initial conditions of the discharge.

In the case of the hyperbolic roots

$$q = e^{-at}\left(c_1\,e^{\sqrt{-1}\,i\,nt} + c_2\,e^{-\sqrt{-1}\,i\,nt}\right).$$

Let

$$c_1 = c\,(\cosh\varphi + \sqrt{-1}\,i\sinh\varphi),$$

and

$$c_2 = c\,(-\cosh\varphi + \sqrt{-1}\,i\sinh\varphi);$$

then

$$q = \sqrt{-1}\,i\,c\,e^{-at}\left(\cosh\varphi\,\frac{e^{\sqrt{-1}\,i\,nt} - e^{-\sqrt{-1}\,i\,nt}}{2}\right.$$

$$\left. + \sinh\varphi\,\frac{e^{\sqrt{-1}\,i\,nt} + e^{-\sqrt{-1}\,i\,nt}}{2}\right)$$

$$= \sqrt{-1}\,i\,2\,c\,e^{-at}\sinh\,(nt + \varphi),$$

and by dropping $\sqrt{-1}\,i$ and writing A for $2\,c$,

$$q = A\,e^{-at}\sinh\,(nt + \varphi).$$

Were conjugate hyperbolic angles taken for the arbitrary constants, the horizontal projection would be obtained, involving $\cosh\,(nt + \varphi)$ in which case the initial current could not be zero. Either projection satisfies the differential equation, but it is only the former which satisfies the initial condition that there is no current at the beginning.

The meaning of these solutions is illustrated by Figs. 3 and 4. Fig. 3 represents the circular case. O P multiplied by c represents c_1, and O P¹ multiplied by c represents c_2; O Q multiplied by $c\,e^{-at}$ represents the first circular solution and O Q¹ multiplied by the same quantity represents the supplementary circular solution. The multiples of O Q and O Q¹ are compounded, their resultant being $2\,c\,e^{-at}$ of O M which represents sin $(n\,t + \varphi)$.

In the hyperbolic case (Fig. 4), O P multiplied by c represents c_1, and O P^1 multiplied by c represents c_2; O Q multiplied by $c\,e^{-at}$ represents the first hyperbolic solution, and O Q^1 multiplied by the same ratio represents the supplementary hyperbolic solution. The multiples of O Q and O Q^1 are compounded, their resultant being $2\,c\,e^{-at}$ of O M, which represents the sine of the hyperbolic angle $n\,t + \varphi$.

By differentiation we deduce the solution for the current; let it be denoted by I. As $I = \frac{d\,q}{d\,t}$

$$I = -A\,e^{-at}\,[a \sinh (nt + \varphi) - n \cosh (nt + \varphi)]$$

$$= -A\,\sqrt{a^2 - n^2}\,e^{-at}\left[\frac{a}{\sqrt{a^2 - n^2}} \sinh (n\,t + \varphi) - \frac{n}{\sqrt{a^2 - n^2}} \cosh (nt + \varphi)\right]$$

$$= -A\,\sqrt{a^2 - n^2}\,e^{-at} \sinh \left(nt + \varphi - \tanh^{-1} \frac{n}{a}\right).$$

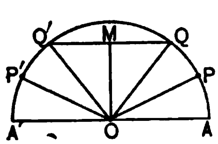

FIG. 3.

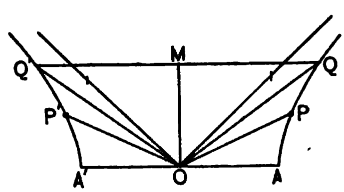

FIG. 4.

Thus the charge is in advance of the current by the hyperbolic angle whose tangent is $\frac{n}{a}$, which is the hyperbolic angle at which both q and I have their maximum value. The same proposition applies, *mutatis mutandis,* to the oscillating discharge.

Writers on this subject call $\frac{1}{a}$ the time constant for an exponential discharge, and $\frac{1}{-a + \sqrt{a^2 - b}}$ and $\frac{1}{-a - \sqrt{a^2 - b}}$ the time constants for the non-oscillating discharge. But from the above presentation of the subject it is evident that $\sqrt{a^2 - b}$ is the analogue of $\sqrt{b - a^2}$ in the circular case. There it means the angular velocity of the auxiliary circular motion; so here it means the angular velocity of the auxiliary equilateral-hyperbolic motion. In the oscillating case $\frac{2\pi}{\sqrt{b-a^2}}$ gives the period; in the non-oscillating case $\frac{2\pi}{\sqrt{a^2-b}}$ gives the hyperbolic period. By the hyperbolic period is meant the time occupied by the radius-vector of the equilateral hyperbola of unit semi-axis to sweep out twice the area of the circle of unit radius. This definition of *period* applies to the circular case also.

The function $A \sin(nt + \varphi)$ represents the vertical projection of a uniform circular motion of amplitude A, angular velocity n, and epoch φ. Similarly the function $A\, e^{-at} \sin(nt + \varphi)$ represents (Fig. 5) the vertical projection of the circular spiral motion of the point P having angular velocity n, epoch φ and logarithmically decreasing amplitude $A\, e^{-at}$. In the same manner the

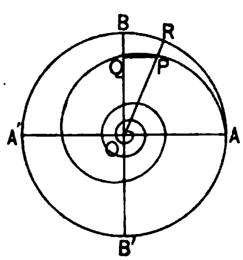

Fig. 5.

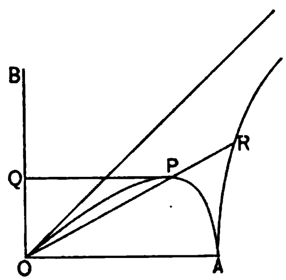

Fig. 6.

function $A\, e^{-at} \sinh(nt + \varphi)$ represents (Fig. 6,) the vertical projection of the hyperbolic spiral motion of the point P having hyperbolic angular velocity n, epoch the hyperbolic angle φ, and amplitude $A\, e^{-at}$. It will be observed that this spiral is convergent, for n is less than a.

By putting in the conditions that $I = 0$ and $q = Q$ when $t = 0$, we obtain

$$\varphi = \tanh^{-1} \frac{n}{a}\,, \text{ and } A = Q\, \frac{\sqrt{a^2 - n^2}}{n}$$

consequently

$$q = Q\, \frac{\sqrt{a^2 - n^2}}{n}\, e^{-at} \sinh\left(nt + \tanh^{-1} \frac{n}{a}\right)$$

and

$$I = -\, Q\, \frac{a^2 - n^2}{n}\, e^{-at} \sinh nt.$$

These curves have a maximum value when the angle is $\tanh^{-1}\frac{n}{a}$; hence when $t = o$ and $t = \frac{1}{n}\tanh^{-1}\frac{n}{a}$ respectively. They have a point of contrary flexure, when the angle is $2\tanh^{-1}\frac{n}{a}$; hence when $t = \frac{1}{n}\tanh^{-1}\frac{n}{a}$ and $t = \frac{n}{2}\tanh^{-1}\frac{n}{a}$ respectively. The properties of either curve are given by the general equation

$$\frac{d^m q}{dt^m} = (-1)^m Q \frac{\sqrt{(a^2 - n^2)}^{m+1}}{n} e^{-at} \sinh\left[nt - (m-1)\tanh^{-1}\frac{n}{a}\right]$$

corresponding to

$$\frac{d^m q}{dt^m} = (-1)^m Q \frac{\sqrt{(a^2 + n^2)}^{m+1}}{n} e^{-at} \sin\left[nt - (m-1)\tan^{-1}\frac{n}{a}\right]$$

in the oscillating case.

The nature of the curves for the charge and the current in the non-oscillating case has not been plain to some electricians of high authority. In the first volume of his work, "Alternating Current Transformer," page 379, Professor Fleming represents the current graphically by an exponential curve, which is far from representing the current correctly. In the first volume of his "Lecons sur l'Electricité," page 256, Professor Gerard represents the charge by an exponential curve which has no maximum at the beginning; and the same representation is given by Professors Jackson in appendix C of their "Alternating Currents." The curves are correctly represented graphically by Doctors Bedell and Crehore in their "Alternating Currents," and by Professor Webster in his "Theory of Electricity and Magnetism."

We deduce the solution for the transition case by means of the principle that in form it must agree with what is common to the two general solutions. Now for the hyperbolic case

$$q = A\, e^{-at}\left[nt + \varphi + \frac{(nt+\varphi)^3}{3!} + \right],$$

and for the circular case

$$q = A\, e^{-at}\left[nt + \varphi - \frac{(nt+\varphi)^3}{3!} + \right];$$

hence for the transition case

$$q = A\, e^{-at}\, (\varphi + n\, t).$$

As $A\,\varphi$ is represented by a length, and $A\,n$ by a linear velocity, let them be denoted by the constants $2\,c$ and $2\,v$. Then $q = 2\,e^{-at}\,(c + v\,t)$.

In the case of the horizontal projections the only common part is the first term of the series, namely 1; hence b denoting an arbitrary length, we have $2\,e^{-at}\,b$ for that projection. Hence the primary form of the solution of the differential equation in the transition case is

$$q = e^{-at}\,\{ [b + i\,(c + v\,t)] + [-b + i\,(c + vt)] \}.$$

This is represented in Fig. 7, which is the transition between Figs. 3 and 4. O P represents $b + i\,c$, and O P^1 represents $-b + i\,c$; O Q represents $b + i\,(c + v\,t)$ and O Q^1 represents $-b + i\,(c + vt)$; O M represents half of the resultant of O Q and O Q^1.

By putting in the conditions that $I = 0$ and $q = Q$ when $t = 0$, we obtain

$$q = Q\, e^{-at}\, (a\,t + 1)$$

$$\text{and } I = -\,Q\, e^{-at}\, a^2\, t.$$

The general differential co-efficient is

$$\frac{d^m q}{d\,t^m} = (-1)^m\, Q\, e^{-at}\, a^m \left[a\,t - (m - 1) \right].$$

Hence q is a maximum when $t = o$, and has a point of contrary flexure when $t = \frac{1}{a}$; and I has a maximum when $t = \frac{1}{a}$ and a point of contrary flexure when $t = \frac{2}{a}$. Thus we see that 1 takes the place of $\tan^{-1}\frac{n}{a}$ or $\tanh^{-1}\frac{n}{a}$, and that a takes the place of n.

Fig. 8 is the transition between Figs. 5 and 6. The point R describes a uniform motion along the straight line; O P is O R diminished at a uniform geometrical rate, O Q is the vertical projection of O P. The path of P is perpendicular to O A at the point O, whereas in the hyperbolic case it makes an angle of 45°.

If attention is restricted to real roots, it is difficult to see why the transition solution is not of the form $q = A\,e^{-at}$, nor is the matter made very clear in treatises on Differential Equations.

The preceding investigation throws new light on the theory of the quadratic equation. The current theory may be stated as follows: A quadratic equation has either two real roots, or two imaginary roots, the separating case being when the roots are equal. According to the results of the preceding investigation, the theory should be stated as follows: So far as real roots are concerned, a quadratic equation has either two such roots, or else none, the separating case being where they are equal. The two general cases are the real and the impossible. As regards complex roots, a quadratic equation has either two conjugate hyperbolic roots, or else, two conjugate circular roots, the separating case being where they are straight-line. Consider the quadratic equation $x^2 + 2\,ax + b = 0$. If a^2 is greater than b, the roots are hyperbolic, and

$$x_1 = -a + \sqrt{-1}\, i\, \sqrt{a^2 - b}$$
$$x_2 = -a - \sqrt{-1}\, i\, \sqrt{a^2 - b}$$

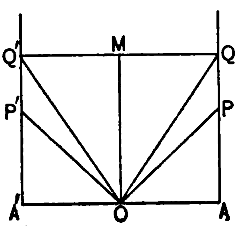

Fig. 7.

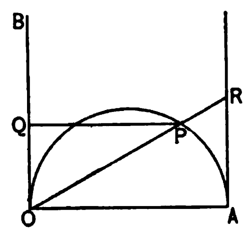

Fig. 8.

If we substitute either root in the equation, we shall find, just as in the case of the circular roots, that the terms which do not involve i cancel one another, and likewise the terms which do involve i. The equation is doubly satisfied by the independent vanishing of the two parts.

The preceding investigation has an important bearing on the theory of the complex quantity, a theory which lies at the foundation of algebraic analysis. The eminent mathematician Cayley maintained that the complex quantity $a + ib$ is the most general magnitude considered by algebra, and that were it fully investigated the science would become *totus teres atque rotundus*. The current doctrine among mathematicians is thus stated in a recent able work on alternating currents, where from the nature of the subject the circular complex quantity is a fundamental idea:

"Within the range of algebra no further extension of the system of numbers is necessary or possible, and the most general number is $a + ib$, where a and b can be integers or fractions, positive or negative, rational or irrational." [1]Let the question be limited to the algebra of the plane although that is in truth an arbitrary restriction, for spherical trigonometrical analysis is as much algebra as is plane trigonometrical analysis. The preceding investigation shows that the ordinary complex quantity is only one-half of the whole subject of plane algebra; for parallel with the circular complex quantity we have a hyperbolic complex quantity, and for every theorem about the former there is an analogous theorem about the latter. If the one is within the domain of algebra, so is the other. Here we have another instance of the danger involved in predicating *impossible.*

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1. Steinmetz, "Alternating Current Phenomena," p. 405.

DISCUSSION AT ELIOT, ME., JULY 27, 1897.

MR. CHAS. P. STEINMETZ:—I have been very much interested in this paper of Dr. Macfarlane's, since it offers a common method of calculation for the two different classes of phenomena which can take place, if a condenser discharges through a circuit of resistance and inductance. Such a condenser discharge can either be an oscillating current, that is, a current similar to an alternating current, but gradually decreasing in amplitude, or it may be a steady discharge, that is a gradual dying out without reversal. The former is the case if the resistance of the circuit is below, the latter if it is beyond a certain critical value.

The usual method of investigation of these phenomena leads to two different functions, an exponential function for the steady discharge which becomes imaginary if the resistance is below the critical value, and a trigonometric function for the oscillating discharge which becomes imaginary if the resistance is above the critical value.

Both these two forms of discharge are apparently entirely different from each other, nevertheless in reality they are of the same nature and gradually change into each other as you will best see by considering a mechanical analogon, as a pendulum in motion. A pendulum, when set in motion in air, will make a lesser or larger number of oscillations with gradually decreasing amplitude, until it comes to a standstill. The rapidity of decrease of oscillations depends upon the resistance of the medium in which it moves, thus for instance in a more resisting medium as in water, the amplitude of oscillation of the pendulum will decrease much faster than in air and it will come to rest very shortly. In a still more viscous medium the oscillations will die out still more rapidly and only very few oscillations take place and ultimately only a half oscillation, that is, the pendulum will be brought to rest by the resistance of the medium without ever passing over the position of equilibrium. The latter case is analogous to the steady discharge.

It is very gratifying to see in the paper these two sides of the same phenomenon of condenser discharge, represented by the same symbolism.

There is, however, one sentence in the end of the paper in reference to a statement which I made once regarding the position of the complex imaginary quantity in algebra. I am sorry to say that I cannot agree with Dr. Macfarlane on this point, but have to maintain my former position.

All the eminent mathematicians of modern times as far as they have taken position at all, have considered the complex imaginary quantity as the most general algebraic quantity, and that there is no further extension of algebra possible outside of the complex imaginary quantity. It is obvious that these mathematicians cannot possibly have been mistaken.

In opposition hereto, Dr. Macfarlane takes the stand that there are other complex imaginary quantities possible outside of the complex imaginary one, and even offers here in the paper a complex imaginary or hyperbolic quantity outside of the complex quantity of pure mathematics. It is evident that there must be a mistake somewhere, and we can discover the mistake by looking on page 165 of Dr. Macfarlane's paper, where he introduced this hyperbolic complex imaginary quantity. He says there in the second line, "If we inquire into the geometrical meaning of the $\sqrt{-1}$ here appearing, we shall find that it means a quadrant of turning round the axis perpendicular to the plane of reference," then he introduces the hyperbolic quantity. This shows the source of the mistake. The quantity introduced here by Dr. Macfarlane is a vector symbolism, that is the symbolic representation of a plane vector, or in other words, of a geometrical relation, similar as for instance the system of quaternions is a space vector symbolism. A vector symbolism is not yet an algebraic quantity, for instance the quarternions are not algebraic quantities.

Undoubtedly the number of possible vector symbolisms is unlimited, the number of algebraic quantities is however limited.

We have thus to see what is the meaning of an algebraic quantity.

An algebraic quantity is a quantity with which we can operate by the common rules of algebra, that is, multiply, divide, solve equations, etc.

There is one fundamental principle underlying all algebraic operation, that is the principle that if a product is zero one of the factors must be zero.

Even the simplest calculations are based on this principle.

Take for instance the case that you have measured a resistance r and find,

$$5\,r = 20$$

everybody will say herefrom,

$$r = 4.$$

Explicitly the operations were,

$$\begin{aligned} &5\,r = 20 \\ &5\,r - 20 = 0 \\ &5\,(r - 4) = 0 \\ &r - 4 = 0 \\ &r = 4 \end{aligned}$$

As you see in deriving this result $r = 4$ from $5\,r = 20$, a result which everybody would take for granted immediately, we have made use of the fundamental principle of algebra by cancelling with 5, that is assuming that if the product $5\,(r - 4) = 0$ and the one factor 5 is not zero, the other factor $r - 4$ must be zero. Thus if this principle does not hold, the calculation would be erroneous, or in other words, every possibility of calculation would cease, and it is erroneous for quantities which are not algebraic, as for instance for the hyperbolic vector quantity introduced by Dr. Macfarlane.

Let us see now how the complex imaginary quantity stands with regard to this fundamental principle of algebraic quantities.

Let $a + j\,b$ be a complex imaginary quantity and $x + j\,y$ another complex imaginary quantity. Let their product be zero, that is,

$$(a + j\,b)\,(x + j\,y) = 0$$

If now $a + j\,b$ is a quantity differing from zero, the above mentioned fundamental principle of algebra requires that we can cancel the equation with $a + j\,b$ and thus get,

$$x + j\,y = 0$$

that is in other words, if $(a + j\,b)\,(x + j\,y) = 0$ and $a + j\,b$ differs from zero, $x + j\,y$ must be zero, if the complex quantity is an algebraic quantity.

From $(a + j\,b)\,(x + j\,y) = 0$ it follows:

$$(a\,x + j^2\,b\,y) + j\,(a\,y + b\,x) = 0$$

or since $j^2 = -1$ by definition of the imaginary unit, it is

$$(a\,x - b\,y) + j\,(a\,y + b\,x) = 0$$

since, however, j is a different kind of unit, either of the two terms of the last equation must be zero, that is,

$$a\,x - b\,y = 0$$
$$a\,y + b\,x = 0$$

Since a and b differ from zero, we can eliminate a and b from these two symmetrical equations and get the result,

$$x^2 + y^2 = 0$$

This condition can be fulfilled only by

$$x = 0$$
$$y = 0$$

that means $x + j\,y = 0$.

Thence from the equation,

$$(a + j\,b)\,(x + j\,y) = 0$$

it follows,

$$x + j\,y = 0$$

that is the complex imaginary quantity is an algebraic quantity and can be treated as such.

Take, however, the hyperbolic complex quantity introduced by Dr. Macfarlane.

The hyperbolic imaginary unit of his is, $\sqrt{-1}\,\beta^{\frac{\pi}{2}}$ or $i\,\sqrt{-1}$.

For brevity, we may denote: $i\,\sqrt{-1} = k$. It is defined by,

$$k^2 = +1$$

assuming again the product of two such complex quantities equal zero:

$$(a + k\, b)\,(x + k\, y) = 0$$

it follows

$$(a\, x + k^2\, b\, y) + k\,(a\, y + b\, x) = 0$$

or since $k^2 = +1$ and the two terms of the last equation are of different nature, it follows,

$$a\, x + b\, y = 0$$
$$a\, y + b\, x = 0$$

or, eliminating a and b, we get,

$$x^2 - y^2 = 0$$

that is,

$$x = \pm\, y$$

This means, if $x = \pm\, y$, the product $(a + k\, b)\,(x + k\, y)$ can be zero without either factor being zero.

Since x and y differ from zero it follows analogously, as condition of zero value of the above given product, that,

$$a = \mp\, b$$

that is, if in two hyperbolic complex imaginary quantities the coefficient of the real term equals the coefficient of the imaginary term, but the signs of the imaginary components are opposite, the product of these two hyperbolic complex quantities is zero without either factor being zero.

$$(a \pm k\, a)\,(x \mp k\, x) = 0$$

Furthermore it follows that the hyperbolic complex imaginary quantity of this paper is not an algebraic quantity, and that equations in such quantities cannot be handled by the laws of algebra, for instance, cannot be cancelled by a constant factor without being liable to see an equality changed into an inequality. That means the hyperbolic imaginary quantity, while a vector symbolism, is not an algebraic quantity, and my statement that the complex imaginary quantity is the only and most complete algebraic quantity still holds.

DR. A. E. KENNELLY:—I am sorry that Dr. Macfarlane is not with us, because we might expect an animated discussion on the points Mr. Steinmetz has raised. There are several matters of great interest in this paper. I will only venture to call attention to two of them.

We all know that if a plane cuts a right cone at right angles to its axis, the curve of intersection is a circle. If the cutting plane is tilted, the circle becomes an ellipse; until, when the plane is parallel to one side of the cone the curve becomes a parabola; and, finally when the plane is tilted beyond this position the curve is a hyperbola.

The results pointed out by the author may be expressed as follows: When a perfect condenser discharges through a perfectly conducting circuit, *i. e.*, a resistanceless circuit, containing inductance, the discharge is oscillatory without damping, and may be represented by the motion of a point in an ellipse with equiangular velocity around the centre, the angle being measured elliptically; or, by the motion of a point in a circle with equiangular velocity, as a particular case of the ellipse.

When the discharging circuit is not perfectly conducting, but offers a resistance, such that its magnetic time-constant is greater than one-fourth of the static time-constant, the discharge, no longer undamped, may be represented by equiangular velocity in an ellipse—with the circle as a particular case—accompanied by a logarithmic shrinking of the radius vector. The trace of the point's motion is therefore a logarithmic spiral, or an orthogonal projection of the same on an inclined plane, *i. e.*, a logarithmic elliptical spiral.

When the discharging circuit has so much resistance that the magnetic time-constant is greater than one-fourth of the static time-constant, the cutting plane of the cone must be tilted from the elliptic curve to the hyperbolic curve. The discharge of the condenser may now be represented by equiangular velocity in the hyperbola, a rectangular hyperbola as a particular case, accompanied by logarithmic shrinking of the radius vector. The trace of the point's motion is therefore a logarithmic hyperbolic spiral.

When the discharging circuit has critical resistance, so that the magnetic time constant l/r is one-fourth of the static time constant or $c\,r/4$, the cutting plane must occupy the intermediate position, and the curve becomes a parabola. The discharge of the condenser may now be represented by equiangular velocity in the parabola, accompanied by logarithmic shrinking of the radius vector. The trace of the point's motion is therefore a logarithmic parabolic spiral.

I do not mean that the above statements are all in the paper before us, although some of them are there. I mean that these are conclusions to which the very interesting treatment in the paper appears to lead.

The second point is of less interest to us as electricians, it is at present almost wholly of mathematical interest. The square root of minus unity was originally called imaginary, because it was regarded as a logical reduction to an impossible or unusual result. Later it became readily interpreted geometrically as the operator which rotates a line counter-clockwise in a plane through a right angle. No other interpretation has hitherto been given to this symbol so far as I know. Dr. Macfarlane now points out that it is also capable of a new meaning in which it is not a versor but a scalar.

Mr. Steinmetz:—I may add that whatever criticisms I have made does not apply to the particular application of hyperbolic imaginary quantities made by Dr. Macfarlane, since in his paper these complex quantities are never used as products, as for instance, common complex imaginary quantities are used when deriving E. M. F. as product of current, impedance, etc. It is only in this latter case where it becomes essential that the complex quantity is an algebraic quantity.

Dr. F. A. C. Perrine:—There is just one point—in speaking of complex quantities, which are really not the subject of the paper, but nevertheless have been brought up in this discussion—in respect to which I think there is an error that has been commonly committed and has interfered with the general understanding of the complex quantity, and that is the error of considering the introduction of the complex quantity solely, as Mr. Kennelly says, for the geometrical interpretation of a particular mathematical symbol, and that our interpretation of the complex quantity is that if we plot along one axis the value J, and on the other axis the value of the real quantity—we will call it A—than any vector may be represented by a sum which we will call $A + B J$. Now additions or subtractions may be performed entirely geometrically. Let us attempt to obtain a multiplication. Take another vector which we will call $C + D J$, and commonly we refer this to a unit vector in order to obtain multiplication, take this point as the unit vector, then the multiplication of these two quantities is performed by drawing on the first vector a triangle entirely similar to the triangle drawn by joining the end of the unit vector and the second vector. This will give us a third vector which, in the geometrical interpretation, has been sometimes called the product of these two vectors. Now as an actual fact, that is not the product which we can use in our analytical working, for the real product of these two vectors is the product of their scalar values times the cosine of the angle between them, which is an entirely different product from this geometrical form of product. This is an error I have never seen pointed out, though I have noticed more than one writer, shortly after the World's Fair, who began to explain the system, and after reaching geometrical multiplication was compelled to stop, because the saying that this vector representation of the complex quantity is a geometrical method of interpreting the square root of minus one, fails as soon as we accept the geometrical method of multiplication which is the only method of multiplication of the complex vectors which has been at all described in the chapters in the algebras on the complex quantities, and it leads to entirely wrong results, and it shows at once that the true method of dealing with the complex quantities is an algebraic method and not at all a geometric method.

Mr. Steinmetz:—I agree with the last speaker that in multiplying complex imaginary quantities we have to deal algebraically, since in multiplying two complex imaginary quantities geometrically, the product of the two quantities may have no meaning.

In the geometrical multiplication the product of two complex quantities as vectors is a vector again. Wherever the product of the two quantities represented by the complex numbers is not a vector quantity, it is obviously not feasible to represent it in the diagram by a vector, and thus the geometrical multiplication becomes meaningless. This is for instance the case when multiplying current and E. M. F. to derive the power, since power is of double frequency and thus cannot be represented in the polar vector diagram of single frequency.

DR. PERRINE:—That is perfectly true; but at the same time it is not as much a foregone conclusion as you might think; because I was taught to multiply these complex quantities in that way by a man who had learned it from no less a man than Weyerstrass in Germany, and this was given to me as the sum of knowledge in regard to complex quantities, and I think that a great many learners have been confused by exactly that error in the geometrical interpretation of complex quantities.

MR. STEINMETZ:—The geometrical multiplication of complex quantities is obviously just as correct as the algebraical multiplication, wherever the product can be represented geometrically, that is wherever the product is a vector, as for instance when multiplying current with impedance or E. M. F. with admittance. Only where the product of the two vectors is not a vector, as when multiplying current and E. M. F., the geometrical multiplication which gives a vector as product, becomes meaningless.

DR. LOUIS BELL:—It seems to me that this discussion gives a very beautiful example of how not to use certain mathematical symbols. I think we owe to Mr. Steinmetz a great deal for his clever manipulation of the imaginary quantity in the solution of physical problems, but I do not know that anything has been brought out more clearly in this discussion, than the necessity of tying yourself up to your physical conceptions. Whenever you let yourself loose and deal with imaginary quantities which have not a close and precise physical meaning, you are apt to get into trouble. Just so long as you tie yourself up, keep out your sheet anchor, look sharply as to the character of the quantities with which you are dealing, you will derive very valuable results. As soon as you cease remembering that these imaginary quantities are means to an end, that they form a vastly convenient and very precise method of dealing with certain physical things—as soon as you forget that, and start with a free hand to swing your complex quantities for the purpose of seeing what transcendental results you can get, then you are almost ready to get into difficulty. Mr. Steinmetz has given us some beautiful examples of the way in which it should be used, and Dr. Perrine has certainly given one or two examples of the way in which it should not be used. That fact runs all through our physical mathematics; that as soon as you get out of sight of physical interpretations and start out with a free sheet your cruise may bring you to some place which you desire to reach; it may land you in a place from which nothing but an all-wise providence can extricate you.